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APPLICATION OF PINN AND THE METHOD OF DIFFERENTIAL CONSTRUCTION OF ADAPTIVE ONE-DIMENSIONAL COMPUTATIONAL GRIDS

Abstract. This paper presents an approach to constructing adaptive one-dimensional computational grids using the Beltrami equation and Physics-Informed Neural Networks (PINNs). The main focus is on exploring the potential for precise control of grid node density through the control function ω(s), which allows the grid to adapt to the local features of the problem. The Beltrami equation used, being a key component of the method, regulates the distribution of nodes by modifying the function's derivatives depending on the values of the control function. The effectiveness of this approach is demonstrated through examples involving one and two regions of node clustering.

The results showed that the PINN method combined with the Beltrami equation allows for the creation of computational grids with a high degree of adaptation to given conditions, providing detailed modeling in critical regions. This approach has advantages over traditional numerical methods, as integrating physical laws in the grid construction process minimizes numerical errors and improves modeling accuracy. The use of neural networks offers flexibility in model tuning and the ability to account for complex nonlinear dependencies. The discussion of the results highlights the potential of using PINNs for adaptive grid construction in various fields requiring precise and efficient modeling. In conclusion, this study confirms that the combination of the Beltrami equation and PINNs is a powerful tool for adaptive grid construction, opening new possibilities for numerical modeling of complex physical processes.

Key words: PINN, Beltrami equation, Adaptive Mesh Generation, Numerical Methods, Neural Networks, Mesh Optimization, Numerical grids.

1. Introduction

The construction of computational grids plays a key role in numerical modeling and solving problems in mathematical physics. Grids are used to discretize continuous domains, enabling computations through numerical methods such as the finite element method or the finite difference method [1]. The quality of the computational grid significantly affects the accuracy and convergence of the solution: overly coarse elements can lead to significant errors, while overly fine grids may result in excessive computational costs. An optimal grid should adapt to the features of the domain's geometry and the solution's gradients, ensuring a balance between accuracy and efficiency [2]. Various algorithms exist for the automatic generation and adaptation of computational grids, which improve the quality of modeling complex physical processes and minimize computational resources.

In modern numerical modeling methods, several approaches to constructing adaptive computational grids exist, which enhance computational accuracy and the efficiency of problem-solving [3]-[8]. Each method has its own advantages and disadvantages, and the choice of approach depends on the specifics of the problem, accuracy requirements, available computational resources, and the geometry of the domain. The use of adaptive computational grids can significantly improve the quality of numerical calculations and reduce errors in modeling complex physical processes [9].

One of the modern approaches that ensures high accuracy in grid construction is the use of the Beltrami equation, which enables the creation of adaptive grids [10]. The Beltrami equation, originating from complex analysis, describes deformations that preserve angles and control the degree of shape distortion. This equation allows for the construction of grids where the density and structure of cells can be locally adapted according to specified conditions or a metric function. The application of the Beltrami equation in grid generation tasks provides controlled deformation of grid cells, minimizing errors and improving the convergence of numerical solutions.

Physics-Informed Neural Networks (PINNs) combine machine learning with physical laws to solve partial differential equations (PDEs) [11]. In this approach, the network is trained not only on data but also by incorporating physical equations into the loss function. This allows the network to approximate solutions to PDEs while minimizing errors with respect to the given equations and boundary conditions [12]. One of the main advantages of PINNs is their ability to effectively handle problems with complex geometry and account for nonlinear dependencies.

The article [13] examines the application of Physics-Informed Neural Networks (PINNs) for solving the equidistribution method equations used in adaptive grid construction. The equidistribution method ensures uniform node distribution to reduce numerical errors, while PINNs assist in solving the associated equations by minimizing deviations from physical laws [3]. The combination of these approaches allows for the efficient construction of grids that adapt to complex geometries and the physical features of the problem. The author demonstrates that the use of PINNs improves grid construction accuracy and reduces computational costs, making this method promising for modeling complex systems.

The application of PINNs for constructing computational grids enables adaptive changes in node density based on the local features of the problem, which is particularly useful for modeling processes with sharp gradients or localized features [14]. Specifically, for the Beltrami equation, which controls node distribution through a control function, PINNs can be used to efficiently manage grid density [15]. This allows for precise adaptation of the computational grid to the problem's conditions, enhancing modeling accuracy and reducing computational costs, making PINNs a promising tool for the numerical simulation of complex physical processes.

2. Materials and Methods

This study employs the Beltrami equations for constructing adaptive computational grids, enabling control over distortions and adaptation of the grid to the specific features of the problem. The Beltrami equation serves as the primary tool for describing quasiconformal mappings, which minimize shape distortions when transitioning between different spatial regions.

For the numerical solution of the one-dimensional Beltrami equation, used in adaptive grid construction, the following method is applied. The main equation is expressed as:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial s}{\partial \xi}\omega(s)\right) = 0$$

where ξ is the independent variable, $s(\xi)$ is the desired function, and $\omega(s)$ is the weight function that controls the influence of the derivative $\frac{\partial s}{\partial \xi}$. This function $\omega(s)$ plays a role analogous to the Beltrami coefficient in multidimensional problems and determines how the grid density changes.

To solve this equation numerically, the domain $\xi \in [0,1]$ is divided into equal parts, forming a grid with nodes. The derivatives in the equation are approximated using the finite difference method. For example, the derivative of the function s with respect to the variable ξ is approximated by the differences in the function's values at neighboring nodes:

$$\frac{\partial s}{\partial \xi} \approx \frac{s_{i+1} - s_{i-1}}{2\Delta\xi}.$$

Within the equation, an intermediate expression A is introduced, which is defined as the product of the derivative $\frac{\partial s}{\partial \xi}$ and the weight function $\omega(s)$:

$$A = \frac{\partial s}{\partial \xi} \cdot \omega(s).$$

For further calculations, an approximation of the derivative A is required, which is also computed using finite differences:

$$\frac{\partial A}{\partial \xi} \approx \frac{A_{i+1} - A_{i-1}}{2\Delta\xi} = 0.$$

This equation connects the values of the function and their derivatives at the grid nodes, allowing the construction of a system of linear equations that can be solved numerically. Standard methods, such as the Thomas algorithm for tridiagonal matrices or iterative methods, can be used to solve such a system.

An important step is verifying the convergence of the solution to ensure that the numerical approximation is sufficiently accurate. Ultimately, this algorithm provides a solution that adapts to the local features of the problem, ensuring precise distribution of the grid nodes.

Physics-Informed Neural Networks (PINNs) are a modern method for solving partial differential equations (PDEs) that combines machine learning approaches with physical laws, enabling the accurate modeling of complex systems. The core idea of PINNs is that the neural network is trained not only on data but also on the physical meaning of the problem, which is encoded through equations in the loss function. This approach allows PINNs to approximate the solution to PDEs by minimizing the difference between predicted values and theoretical results encoded in the equations.

In solving the Beltrami equation using PINNs, the network takes coordinates as input and returns the values of a function approximating the solution of the equation. The network is built with several hidden layers and a small number of neurons to have sufficient power to model dependencies in the data, without being overly complex. A crucial part is constructing the loss function, which takes into account both the boundary conditions and the Beltrami equation itself. This is achieved by using automatic differentiation in PyTorch, which allows computing the derivatives of the neural network with respect to the input variables and incorporating them into the equation's expression.

Let us consider the one-dimensional Beltrami equation described earlier:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial s}{\partial \xi}\omega(s)\right) = 0,$$

where $\omega(s)$ is the weight function controlling the local adaptation of the grid. To solve this equation using PINNs, we use a neural network that approximates the function $s(\xi)$. First, a loss function is defined based on the equation's expression and the boundary conditions. For instance, the derivatives from the network that correspond to $\frac{\partial s}{\partial \xi}$ and $\frac{\partial A}{\partial \xi}$, where $A = \frac{\partial s}{\partial \xi} \cdot \omega(s)$, are computed automatically and compared to zero as the target of the loss function. Thus, minimizing the loss function leads to the solution of the equation over the entire domain.

The code developed as part of this work implements a PINN to solve the one-dimensional Beltrami equation using PyTorch. The neural network is designed with 10 hidden layers, each containing 5 neurons, and the sigmoid activation function is applied after each layer to capture nonlinearity. The Adam optimization algorithm is used to update the network parameters during each iteration based on the gradient of the loss function, which accounts for both the boundary conditions and the equation itself. The boundary loss is computed by measuring the difference between the network's output and the expected values at the boundaries, while the PDE loss is based on the expression for $\frac{\partial A}{\partial \xi}$.

Once the training is complete, the network can predict the values of the function $s(\xi)$, corresponding to the solution of the Beltrami equation. The results can be visualized in 3D plots to evaluate the accuracy of the predictions.

In summary, the code developed in this study demonstrates that PINNs can accurately and efficiently solve complex grid construction problems by fully incorporating physical laws into the training process, expanding the potential of numerical modeling.

3. Results

This study presents an approach for constructing adaptive one-dimensional computational grids using the Beltrami equation and Physics-Informed Neural Networks (PINNs). The primary focus was on analyzing the effect of the control function $\omega(s)$ on the grid node distribution. The simulation results demonstrated that the combination of the Beltrami equation and neural networks effectively controls node density, allowing the grid to adapt to specific regions of interest.

In the initial phase of the study, a scenario was examined where the control function $\omega(s)$ was concentrated around the coordinate 0.3. As shown in Figure 1, the grid nodes cluster near this point, which aligns with the specified control function $\omega(s)$. This function takes into account the contribution of a term centered at 0.3 with a clustering parameter of $\alpha_0 = 0.015$. The shape of the function minimizes numerical errors in the region with a higher node density, which is particularly important for modeling processes with sharp gradients or localized features. This clustering improves the model's resolution in critical regions, as confirmed by the uniform distribution of nodes around the specified coordinate, while the grid remains less dense in other areas.

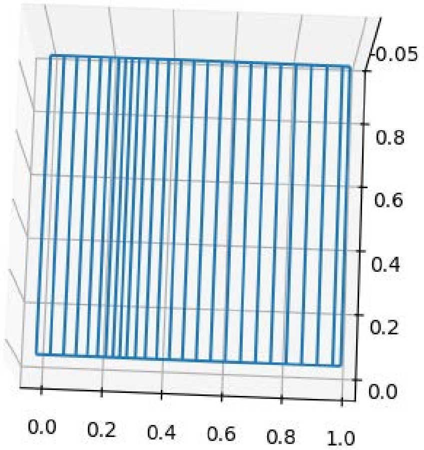

Figure 1 – Grid clustering near coordinate 0.3

In the second phase, the grid was adapted to account for two clustering regions specified by a control function centered at coordinates 0.3 and 0.7. As seen in Figure 2, the grid shows pronounced clustering near both coordinates. The control function in this case is the sum of two modifying terms, each creating its own clustering zone. Consequently, the grid adapts to both regions, increasing the local density of nodes. This approach enables efficient modeling of processes that require detailed focus in multiple areas simultaneously, such as when multiple zones with steep gradients are present. The resulting grid confirms that the combination of PINNs with the Beltrami equation and an appropriately chosen control function can successfully adapt the grid to solve problems with various localized features.

Discussing the results, it is important to highlight that grid adaptation using the Beltrami equation and PINNs offers several advantages over traditional grid generation methods. First, this approach allows for the seamless integration of physical laws into the grid construction process, enhancing both the accuracy and stability of the solution. In this case, the Beltrami equation serves as a tool to control node density, enabling local grid adaptation based on the values of the control function. This is particularly valuable for problems with heterogeneous conditions, where high resolution is needed in specific regions. Second, the use of neural networks provides flexibility in selecting model parameters and allows for capturing complex relationships between grid nodes and the control function.

However, despite the successful application of the method for problems with predefined clustering regions, several limitations of the approach should be considered. In particular, the sensitivity of the results to the choice of parameters for the control function $\omega(s)$ can affect the node distribution, requiring precise tuning for each specific problem. This may complicate the use of the method in problems with dynamically changing conditions, where the parameters of $\omega(s)$ would need to adapt in real-time. Another challenge is the high computational cost associated with training the neural network, especially when using complex architectures and a large number of nodes.

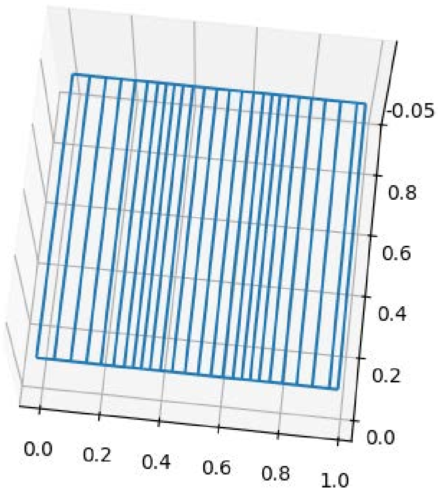

Figure 2 – Grid clustering near coordinates 0.3 and 0.7

Moreover, an important direction for future research is the analysis of the method's stability and accuracy with different types of control functions. In this study, simple functions with one or two clustering peaks were used, but there are situations where a more complex form of $\omega(s)$ may be required, such as a smooth variation in node density across the entire domain. Investigating such functions could open new possibilities for applying the method in various scientific and engineering fields, including aerodynamics, hydrodynamics, and the modeling of complex physical processes.

Thus, the results of this work demonstrate the potential of using the Beltrami equation and PINNs for adaptive grid construction, showing the high efficiency of the method in problems with localized features. This opens up prospects for further development of the approach, aiming to improve grid adaptation in the context of complex and dynamically changing physical processes, which could significantly broaden the method's application in engineering and scientific practice.

5. Conclusions

This study examined an approach for constructing adaptive one-dimensional computational grids using the Beltrami equation and Physics-Informed Neural Networks (PINNs). The main goal of the research was to explore the potential of this method for precise control of grid node density, which is crucial when modeling processes with sharp gradients or localized features. The results showed that the combination of the Beltrami equation with PINNs enables effective grid adaptation to specified regions of interest by modifying the control function $\omega(s)$. This approach demonstrates high flexibility in grid configuration, which is especially important in tasks that require localized node clustering to enhance modeling resolution.

An important aspect of this work was the confirmation that neural networks can successfully integrate physical principles into the grid adaptation process. By using the Beltrami equation, which governs node distribution through the control function, a balance is achieved between the need for detailed representation in certain areas and the overall numerical stability and efficiency of the calculations. It was shown that the PINN method, due to its ability to approximate complex nonlinear dependencies, enables the construction of adaptive grids that are better suited to the specifics of the modeled processes compared to traditional numerical approaches.

The conclusion of this work highlights the promising potential of PINNs for grid construction tasks, particularly in the context of the Beltrami equation. The method demonstrated its advantages in controlling grid density and showed the ability to create highly adaptive grid structures that can be effectively applied to a wide range of scientific and engineering problems. At the same time, the results indicate the need for further research focused on optimizing the control function parameters and developing approaches for automatic grid adaptation in the context of changing physical conditions of the problem. Additionally, further exploration of the method's stability and accuracy for more complex multidimensional cases and dynamically changing systems will be an important step in expanding the application of PINNs to adaptive grid construction.

Thus, the conducted study confirms that the use of the Beltrami equation and PINNs for adaptive grid construction is a promising direction that opens new possibilities for numerical modeling of complex physical processes. The development of this approach will improve the quality of simulations and significantly expand the application of adaptive grids in various high-tech fields, such as aerodynamics, hydrodynamics, and other areas of engineering and science that require precise and efficient numerical solutions.

Author Contributions

Conceptualization, M.M. and O.T.; Methodology, O.T.; Software, M.M.; Validation, M.M. and O.T.; Formal Analysis, O.T.; Investigation, M.M. and O.T.; Resources, O.T.; Data Curation, M.M. and O.T.; Writing – Original Draft Preparation, O.T.; Writing – Review & Editing, M.M.; Visualization, M.M.; Supervision, O.T.; Project Administration, O.T.

Conflicts of Interest

The authors declare no conflict of interest.

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